Cambridge Assessment Admissions Testing

Test of Mathematics for University Admission

Paper 1 2017 worked answers

Test of Mathematics for University Admission
2017 Paper 1
Worked Solutions

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Contents

Introduction for students

These solutions are designed to support you as you prepare to take the Test of Mathematics for University Admission. They are intended to help you understand how to answer the questions, and therefore you are strongly encouraged to **attempt the questions first** before looking at these worked solutions. For this reason, each solution starts on a new page, so that you can avoid looking ahead.

The solutions contain much more detail and explanation than you would need to write in the test itself – after all, the test is multiple choice, so no written solutions are needed, and you may be very fluent at some of the steps spelled out here. Nevertheless, doing too much in your head might lead to making unnecessary mistakes, so a healthy balance is a good target!

There may be alternative ways to correctly answer these questions; these are not meant to be 'definitive' solutions.

The questions themselves are available on the 'Preparing for the test' section on the Admissions Testing website.

Question 1

We are given the derivative of y, and we want to find y itself, so we need to integrate. We start by expanding the fraction:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2 - \frac{2-3x}{x^3} \\ &= 3x^2 - 2x^{-3} + 3x^{-2}\end{aligned}$$

so we can now integrate term-by-term to get

$$y = x^3 + x^{-2} - 3x^{-1} + c$$

Now as $y = 5$ when $x = 1$, we can substitute to give

$$5 = 1^3 + 1^{-2} - 3 \times 1^{-1} + c = -1 + c$$

so $c = 6$ and the answer is C.

> ***Commentary:*** *It turns out that there is a subtlety in this question which we usually ignore at school level. The question assumes that there is a 'simple' formula for y in terms of x. Since the function is not defined at $x = 0$, it is possible that we have different constants when $x > 0$ and when $x < 0$, so we could have, for example,*
>
> $$y = \begin{cases} x^3 + x^{-2} - 3x^{-1} + 6 & \text{when } x > 0 \\ x^3 + x^{-2} - 3x^{-1} + 8 & \text{when } x < 0 \end{cases}$$
>
> *This may seem quite strange, but it would be a valid solution to the differential equation given in the question.*

Question 2

We expand the brackets, writing everything as powers of x:

$$\begin{aligned}
\mathrm{f}(x) &= \left(\frac{2}{x} - \frac{1}{2x^2}\right)^2 \\
&= \frac{4}{x^2} - 2\frac{2}{x}\frac{1}{2x^2} + \frac{1}{4x^4} \\
&= 4x^{-2} - 2x^{-3} + \frac{1}{4}x^{-4}
\end{aligned}$$

(Take care with the last simplification: writing $\frac{1}{4x^4} = 4x^{-4}$ is a common error!)

Then differentiating gives

$$\mathrm{f}'(x) = -8x^{-3} + 6x^{-4} - x^{-5}$$

and differentiating a second time then gives

$$\mathrm{f}''(x) = 24x^{-4} - 24x^{-5} + 5x^{-6}$$

We can now subsitute $x = 1$ to give

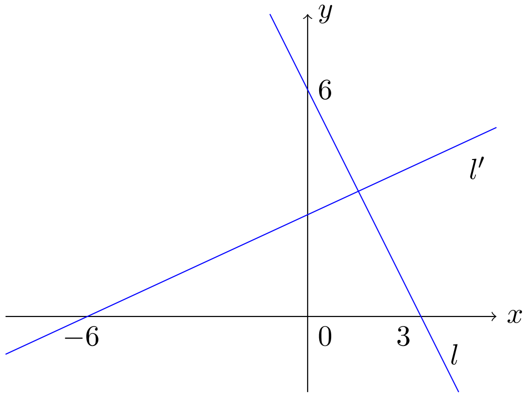
$$\mathrm{f}''(1) = 24 - 24 + 5 = 5$$

so the correct answer is C.

Question 3

It is always worth drawing a sketch for questions like this, to ensure that we know what area is being referred to.

The line l has a negative gradient, and passes through the point $(0, 6)$; it also intersects the x-axis at $(3, 0)$. So the sketch looks something like this:



We see therefore that the required area is a triangle with base length along the x-axis of $3-(-6) = 9$. Thus to find the area we only need to find the y-coordinate of the point of intersection of the two lines.

The equation of the second line, l', can be given by $y - y_1 = m(x - x_1)$: its gradient is the gradient perpendicular to -2, which is $\frac{1}{2}$, and it passes through $(-6, 0)$, so it has equation

$$l' : y - 0 = \tfrac{1}{2}(x - (-6))$$

which simplifies to $y = \frac{1}{2}x + 3$. We can find the coordinates of intersection by solving the equations of l and l' simultaneously. Substituting for y, we get

$$6 - 2x = \tfrac{1}{2}x + 3$$

so $\frac{5}{2}x = 3$, or $x = \frac{6}{5}$. This looks plausible from the sketch. We then get $y = \frac{1}{2}x + 3 = \frac{18}{5}$ (and we can check that the equation of l gives the same y-value). Thus the area of the triangle is

$$\tfrac{1}{2} \times 9 \times \tfrac{18}{5} = \tfrac{81}{5} = 16\tfrac{1}{5}$$

and the correct answer is A.

Question 4

We are dividing a polynomial by a linear expression and asking for the remainder, so it would make sense to use the remainder theorem: if the polynomial $\mathrm{f}(x)$ is divided by $x - a$, then the remainder is $\mathrm{f}(a)$.

In this case, we start with $(3x^2 + 8x - 3)$ and multiply it by $(px - 1)$. This gives our polynomial $\mathrm{f}(x) = (3x^2 + 8x - 3)(px - 1)$. We then divide $\mathrm{f}(x)$ by $(x + 1)$ to get a remainder of $\mathrm{f}(-1) = 24$ (using the remainder theorem).

Substituting $x = -1$ into $\mathrm{f}(x)$ gives

$$\mathrm{f}(-1) = (3 - 8 - 3)(-p - 1) = 8(p + 1)$$

Thus $8(p + 1) = 24$, giving $p = 2$, and the correct answer is B.

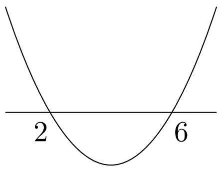
> ***Commentary:*** *We might have been tempted to multiply out the brackets in* $\mathrm{f}(x)$. *But this does not change the value of* $\mathrm{f}(-1)$, *and the remainder theorem does not say anything about the coefficients of* x, x^2 *and so on, so there is no benefit to be gained from expanding* $\mathrm{f}(x)$. *It would only take more effort and increase the chances of making a mistake.*

Question 5

> ***Commentary:*** *The start of this seems like a fairly standard question: find the values of x which satisfy a pair of inequalities. The options offered, though, seem far from standard, so we will have to think carefully once we have found the set S.*

We start by solving the two inequalities separately.

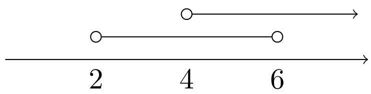
The quadratic factorises as $x^2 - 8x + 12 = (x-2)(x-6)$, so we are solving $(x-2)(x-6) < 0$
We can sketch the graph of $y = (x-2)(x-6)$:



so we see that the first inequality is satisfied when $2 < x < 6$

The second inequality is a linear inequality which immediately simplifies to $2x > 8$ or $x > 4$

We can combine these two inequalities using a number line:



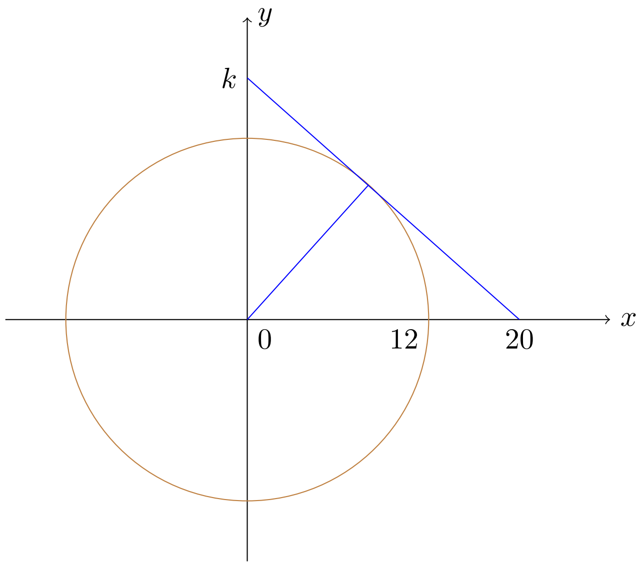
So the set S is given by $4 < x < 6$

Our problem now is that none of the given options says $4 < x < 6$. Most of them are in the form of quadratic inequalities. So we assume that one of the quadratic inequalities probably gives $4 < x < 6$ when we 'solve' it. That, though, seems like a lot of work. We would do better to work in reverse and try to *derive* the quadratic inequality from $4 < x < 6$.

We could think about our earlier sketch, which helped us solve $(x-2)(x-6) < 0$ and gave $2 < x < 6$. If we replace the 2 by a 4, we see that the inequality $(x-4)(x-6) < 0$ will give $4 < x < 6$. We can expand this to get $x^2 - 10x + 24 < 0$, and so the correct answer is C.

Question 6

This calls for a sketch. The circle $x^2 + y^2 = 144$ is centred on the origin and has radius 12:



It's not exact, and it's not drawn to scale, but it's good enough to work with. The blue radius is perpendicular to the tangent line. We are trying to find the value of k.

Approach 1

We could work out the equation of the tangent line. It has gradient m (which we don't know) and passes through $(20, 0)$, so it has equation $y = m(x - 20)$. Since it is tangent to the circle, solving the two equations simultaneously must give one repeated root.

We have $x^2 + y^2 = 144$ into which we can substitute $y = m(x - 20)$ giving

$$\begin{aligned} & x^2 + m^2(x-20)^2 = 144 \\ \iff \quad & x^2 + m^2(x^2 - 40x + 400) = 144 \\ \iff \quad & (m^2+1)x^2 - 40m^2x + 400m^2 - 144 = 0 \end{aligned}$$

We want this to have exactly one root for x, so we need the discriminant to equal zero, giving

$$(-40m^2)^2 - 4(m^2+1)(400m^2 - 144) = 0.$$

We can take out a factor of 4^2:

$$(10m^2)^2 - (m^2+1)(100m^2 - 36) = 0$$

and then expand the brackets to give

$$100m^4 - 100m^4 + 36m^2 - 100m^2 + 36 = 0$$

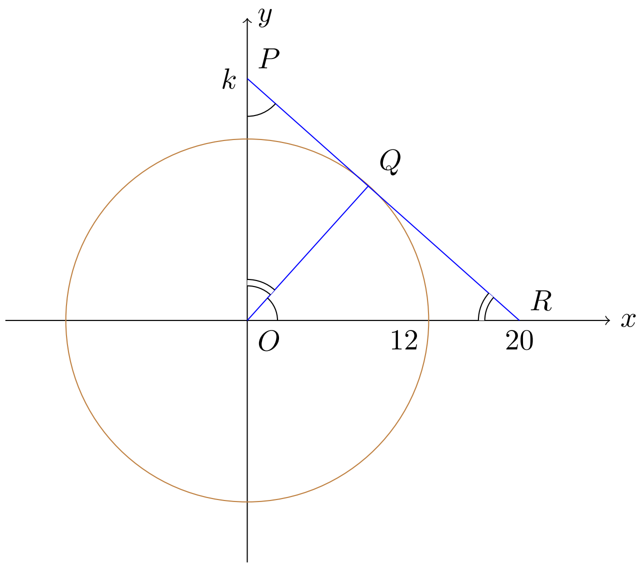
so $64m^2 = 36$, or $8m = \pm 6$. Since $m < 0$ (as the line intersects the positive y-axis), we have $m = -\frac{6}{8} = -\frac{3}{4}$. Thus the tangent line has equation

$$y = -\tfrac{3}{4}(x - 20)$$

Substituting in $(x, y) = (0, k)$ gives $k = 15$, and hence the answer is B.

Approach 2

We might notice from the sketch that we have some similar triangles, since the radius meets the tangent at right angles. If we label all of the vertices and equal angles, we will be able to describe things more precisely:



So the triangles $\triangle OQR$, $\triangle PQO$ and $\triangle POR$ are all similar. We also know the lengths $OR = 20$ and $OQ = 12$ (as it is a radius). By Pythagoras's Theorem, applied to $\triangle OQR$, we can calculate $QR = 16$ (and we might note that $\triangle OQR$ is a 3-4-5 triangle). We can then use the similarity of triangles $\triangle PQO$ and $\triangle OQR$ to find k:

$$\frac{PO}{QO} = \frac{OR}{QR} \implies \frac{k}{12} = \frac{20}{16}$$

giving $k = 15$, as before.

> ***Commentary:*** *You might prefer one or other of these approaches. The second is shorter, but requires some confidence with geometry and recognising similar triangles.*

Question 7

Let the common difference of the arithmetic progression be d and the common ratio of the geometric progression be r. Then we can write the first three terms of each as

$$p,\ q = p + d,\ p^2 = p + 2d \qquad \text{and} \qquad p,\ p^2 = pr,\ q = pr^2.$$

The second term of the geometric progression shows that $r = p$ (as p is non-zero), so the third term gives $q = p^3$

Looking again at the arithmetic progression, using $q = p^3$ now shows that the second and third terms are

$$\begin{aligned} p^3 &= p + d \\ p^2 &= p + 2d \end{aligned}$$

We can eliminate d from these by subtracting twice the first equation from the second, giving $p^2 - 2p^3 = -p$. This rearranges to

$$2p^3 - p^2 - p = 0$$

This factorises as $p(2p^2 - p - 1) = 0$, and then to $p(2p + 1)(p - 1) = 0$. This has the three solutions $p = 0$, $p = -\frac{1}{2}$ and $p = 1$. We were told that $p < 0$, so we must have $p = -\frac{1}{2}$.

Thus $d = p^3 - p = -\frac{1}{8} - (-\frac{1}{2}) = \frac{3}{8}$ and the sum of the first 10 terms of the arithmetic progression is given, using the formula $S_n = \frac{1}{2}n(2a + (n-1)d)$, by

$$S_{10} = \frac{1}{2} \times 10 \left(2 \times \left(-\frac{1}{2}\right) + 9 \times \frac{3}{8}\right) = 5\left(-1 + \frac{27}{8}\right) = \frac{95}{8}$$

and the correct answer is B.

Question 8

Multiplying out the brackets does not look like a particularly useful approach (what would we do with the terms $\cos x$ and $\sin x \cos x$ in the same inequality?), so we will leave the expression in its factored form.

If we temporarily write $a = 1 - 2\sin x$ and $b = \cos x$ we will be able to see the structure of the problem a little more clearly. We want $ab \geqslant 0$, and this will happen when either $a \geqslant 0$ and $b \geqslant 0$, or when $a \leqslant 0$ and $b \leqslant 0$. So we need to solve each of these simpler inequalities.

We have $1 - 2\sin x = 0$ when $\sin x = \frac{1}{2}$, so when $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. We can either sketch the graph of $y = 1 - 2\sin x$ or try values to determine whether $1 - 2\sin x$ is positive or negative in each range. Sketching the graph seems possibly hard, as there are multiple transformations, and it would be easy to make a mistake. So on this occasion, we'll plug in values: $x = 0$ and $x = \pi$ both give $1 - 2\sin x = 1$; while $x = \frac{\pi}{2}$ gives $1 - 2\sin x = -1$. Therefore

$$\begin{aligned} &1 - 2\sin x \geqslant 0 \text{ when } 0 \leqslant x \leqslant \tfrac{\pi}{6} \text{ and when } \tfrac{5\pi}{6} \leqslant x \leqslant \pi \\ &1 - 2\sin x \leqslant 0 \text{ when } \tfrac{\pi}{6} \leqslant x \leqslant \tfrac{5\pi}{6} \end{aligned}$$

We can do similarly for $\cos x$; this time we can think about the graph in our head, and we have

$$\begin{aligned} &\cos x \geqslant 0 \text{ when } 0 \leqslant x \leqslant \tfrac{\pi}{2} \\ &\cos x \leqslant 0 \text{ when } \tfrac{\pi}{2} \leqslant x \leqslant \pi \end{aligned}$$

We can now put everything together. Both expressions are greater than or equal to 0 when $0 \leqslant x \leqslant \frac{\pi}{6}$, and they are both less than or equal to 0 when $\frac{\pi}{2} \leqslant x \leqslant \frac{5\pi}{6}$.
Hence the answer is A.

Question 9

We clearly need to work out the radius of the circle to get anywhere on this problem. We can do this by completing the square. The equation of the circle, $x^2 + y^2 - 18x - 22y + 178 = 0$, becomes

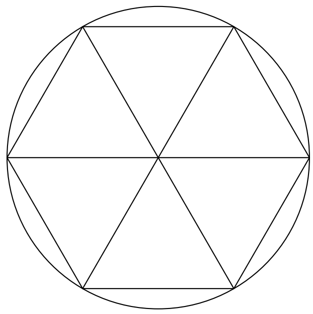
$$(x - 9)^2 - 9^2 + (y - 11)^2 - 11^2 + 178 = 0$$

With a little bit of arithmetic, we can rearrange this to give

$$(x - 9)^2 + (y - 11)^2 = 24$$

so the circle has a radius of $r = \sqrt{24}$

We now need to work out the area of the inscribed regular hexagon. We sketch the hexagon and its diameters:

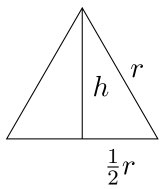


It is clear from the sketch (and even from a not-very-precise sketch) that the hexagon is composed of six triangles, each of which is equilateral and has side length equal to the circle's radius r.

We can work out the area of one of the triangles either using Pythagoras's Theorem or using the formula $\frac{1}{2}ab\sin C$ for the area of a triangle.

The trigonometric formula $\frac{1}{2}ab\sin C$ gives $\frac{1}{2}r^2 \sin 60° = \frac{\sqrt{3}}{4}r^2$. Substituting $r^2 = 24$ then gives each triangle an area of $6\sqrt{3}$, and so the whole hexagon has an area of $36\sqrt{3}$, which is option F.

Alternatively we can proceed using Pythagoras's Theorem. If the height of the triangle is h, then we have:



Therefore $h^2 = r^2 - (\frac{1}{2}r)^2$ giving $h = \frac{\sqrt{3}}{2}r$ and the triangle has area $\frac{1}{2}rh = \frac{\sqrt{3}}{4}r^2$ as before.

Question 10

The gradient of the tangent at x is $f'(x) = -6p^2 + 6px - 3x^2$. Note that p is a constant, and so p^3 differentiates to 0.

The gradient of the tangent where $x = -1$ is therefore $-6p^2 - 6p - 3$, and so the gradient of the normal is the negative reciprocal of this, namely

$$M = \frac{-1}{-6p^2 - 6p - 3} = \frac{1}{3(2p^2 + 2p + 1)}$$

The greatest possible value of this occurs when the denominator of this fraction is as small as possible and positive. We can complete the square to find this:

$$2p^2 + 2p + 1 = 2(p + \tfrac{1}{2})^2 + \tfrac{1}{2}$$

so the minimum of this is $\frac{1}{2}$ (occurring when $p = -\frac{1}{2}$).

We could also have found this using calculus: $\frac{\mathrm{d}}{\mathrm{d}p}(2p^2 + 2p + 1) = 4p + 2$ which is zero when $p = -\frac{1}{2}$, giving a minimum value of $\frac{1}{2}$.

Therefore the greatest possible value of M is $\dfrac{1}{3 \times \frac{1}{2}} = \dfrac{2}{3}$ which is option E.

> ***Commentary:*** *Note that, in this case, the denominator of the fraction expression for M is always positive. If it was instead something like p^2-1, then there would not be a greatest possible value of M, as $p^2 - 1$ could get as close to zero as we like, so M could be as large as we like.*

Question 11

We have $x_1 = 7$, $x_2 = 3$, $x_3 = 1$. It is not at all clear what the pattern here is, so we calculate the next few terms:

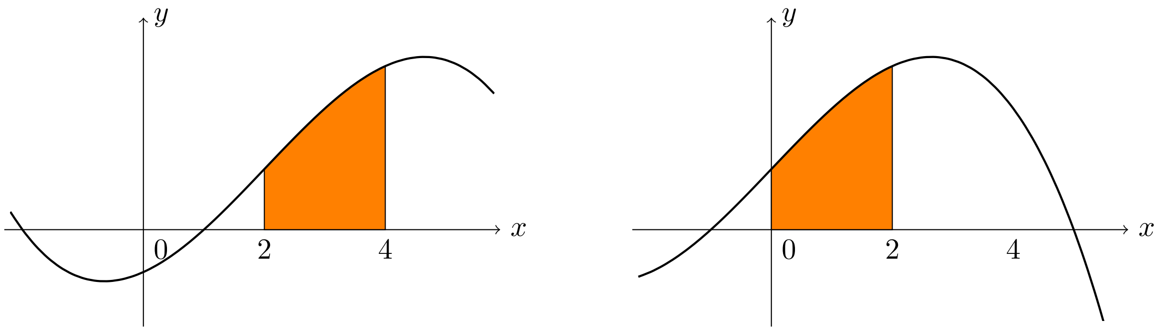
$$x_4 = \frac{23 \times 1 - 53}{5 \times 1 + 1} = \frac{-30}{6} = -5$$
$$x_5 = \frac{23 \times (-5) - 53}{5 \times (-5) + 1} = \frac{-168}{-24} = 7$$

so $x_6 = 3$, $x_7 = 1$, $x_8 = -5$ and so on. Thus the terms repeat in a cycle with length 4, and as $100 = 25 \times 4$, we see that $x_{100} = -5$, which is option A.

Question 12

We have $\int_2^4 \mathrm{f}(x)\,\mathrm{d}x = A$, and the question asks about $\int \mathrm{f}(x+2)\,\mathrm{d}x$ between various limits.

The graph of $y = \mathrm{f}(x+2)$ is the graph of $y = \mathrm{f}(x)$ translated left by 2 units, so we can draw a sketch to see what effect this has on the area between $x = 2$ and $x = 4$:



This means that $\int_2^4 \mathrm{f}(x)\,\mathrm{d}x = \int_0^2 \mathrm{f}(x+2)\,\mathrm{d}x = A$

(If you have learnt about integration by substitution, you can also show this as follows: writing $x = u + 2$ gives $\mathrm{d}x = \mathrm{d}u$, and $u = x - 2$, so $\int_2^4 \mathrm{f}(x)\,\mathrm{d}x = \int_0^2 \mathrm{f}(u+2)\,\mathrm{d}u$)

Therefore the answer is either A or B. We have

$$\int_0^2 [\mathrm{f}(x+2)+1]\,\mathrm{d}x = \int_0^2 \mathrm{f}(x+2)\,\mathrm{d}x + \int_0^2 1\,\mathrm{d}x$$

The first integral equals A, as we have shown, and $\int_0^2 1\,\mathrm{d}x = 2$, so the whole integral equals $A + 2$ and the correct answer is B.

Question 13

The term in x^4 is $\binom{5}{4}a(bx)^4 = 5ab^4x^4$, so the coefficient of x^4 is $5ab^4$

The term in x^2 is $\binom{5}{2}a^3(bx)^2 = 10a^3b^2x^2$, so the coefficient of x^2 is $10a^3b^2$

It follows, since the coefficient of x^4 is 8 times that of x^2, that

$$5ab^4 = 80a^3b^2$$

As a and b are non-zero, we can divide by $5ab^2$ to get $b^2 = 16a^2$. Since both a and b are positive, we can take square roots to get $b = 4a$. (If we didn't know about their signs, we could only say that $b = \pm 4a$.)

So the smallest possible value of $a + b$ occurs when $a = 1$, $b = 4$, giving $a + b = 5$, which is option C.

Question 14

The equations involve 2^x and $2^{2x} = (2^x)^2$, so we write $a = 2^x$; likewise we write $b = 2^y$, and so our equations become:

$$\begin{aligned} a + 3b &= 3 \\ a^2 - 9b^2 &= 6 \end{aligned}$$

The first equation gives $a = 3 - 3b$, which we can substitute into the second equation to obtain

$$(3 - 3b)^2 - 9b^2 = 6$$

which expands to $9 - 18b = 6$, and hence $b = \frac{1}{6}$ and $a = \frac{5}{2}$

Another way to solve these simultaneous equations is to note that the second one can be written as $(a + 3b)(a - 3b) = 6$, using the identity for the difference of two squares. Since $a + 3b = 3$, this gives $a - 3b = 2$, from which we can again deduce a and b.

We can now find x and y:

$$\begin{aligned} x &= \log_2 a = \log_2 \tfrac{5}{2} = p \\ y &= \log_2 b = \log_2 \tfrac{1}{6} = q \end{aligned}$$

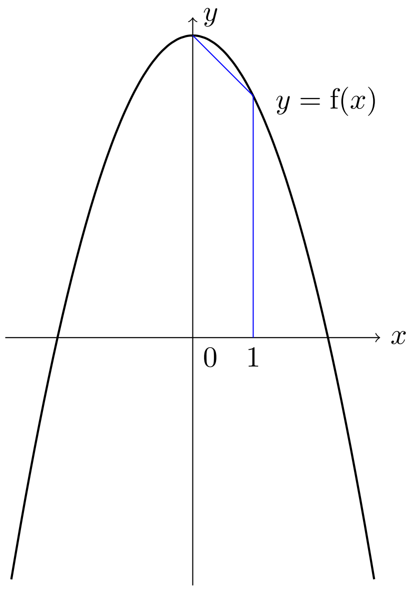
and hence

$$p - q = \log_2 \frac{5}{2} - \log_2 \frac{1}{6} = \log_2 \left(\frac{5}{2} \div \frac{1}{6} \right) = \log_2 15$$

Thus the correct answer is F.

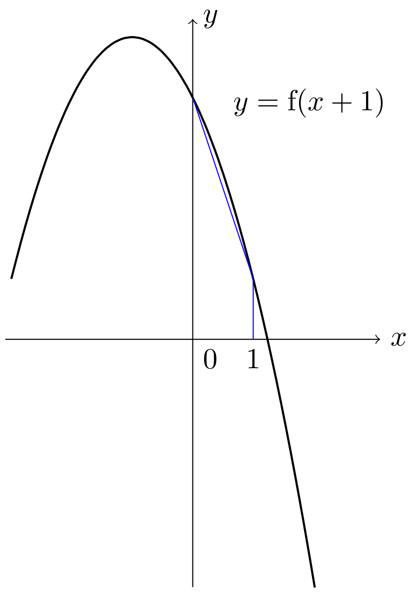
Question 15

We can either proceed by sketching the graph or by using a general theorem about the trapezium rule. We start with sketching. The simplest case of the trapezium rule is where we only have one trapezium, so that is what we will draw.



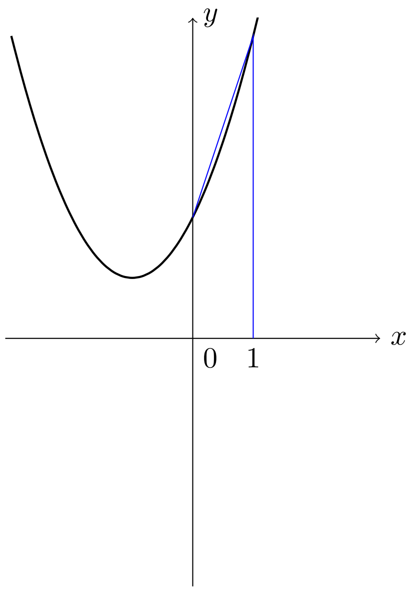
We see here that the trapezium rule underestimates the area of $y = \mathrm{f}(x)$

For the case $y = \mathrm{f}(x + 1)$ the graph of $y = \mathrm{f}(x)$ is translated one unit to the left:



Again, the trapezium rule gives an underestimate.

For the third case, we need to reflect this graph in the line $y = 6$, which gives the following graph:



We see that this time, the trapezium rule gives an overestimate.

So the correct answer is B.

An alternative approach is given by noting that the trapezium rule gives an overestimate of the area under the graph of $y = \mathrm{f}(x)$ if $\mathrm{f}''(x) > 0$ throughout the range and an underestimate if $\mathrm{f}''(x) < 0$ throughout. In our case, $\mathrm{f}''(x) = -4$, so the trapezium rule gives an underestimate for (1). For case (2), $\mathrm{f}''(x+1) = -4$ again, so we still have an underestimate. For the third case, the equation of the graph is $y = 12 - \mathrm{f}(x+1)$, so the second derivative is $-\mathrm{f}''(x+1) = 4$, and therefore we have an overestimate.

Question 16

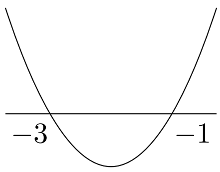
In this question, we consider a function to be increasing when $f'(x) \geq 0$, and decreasing when $f(x) \leq 0$.

> ***Commentary:*** *The classification of increasing and decreasing we use in the options for this question avoids a subtle issue: what we consider a function to be doing (i.e., increasing, decreasing, or neither) at points where $f'(x) = 0$. For this question, though, we don't need to concern ourselves with this as the options all use $\geqslant$ or $\leqslant$. However, we will set out our answer using '$\geqslant$' or '$\leqslant$'.*

We therefore calculate $f'(x)$ and $g'(x)$ and find where each is positive and negative.

$f'(x) = 6x + 12 = 6(x + 2)$, so $f'(x) > 0$ when $x > -2$ and $f'(x) < 0$ when $x < -2$.

$g'(x) = 3x^2 + 12x + 9 = 3(x^2 + 4x + 3)$. We can find where this is positive and negative by sketching the graph: $x^2 + 4x + 3 = (x + 1)(x + 3)$, giving the sketch of $g'(x)$:



so $g'(x) > 0$ when $x < -3$ or $x > -1$, and $g'(x) < 0$ when $-3 < x < -1$.

We now want one of $f'(x)$ and $g'(x)$ to be positive and the other negative.

We have $f'(x) > 0$ and $g'(x) < 0$ when $x > -2$ and $-3 < x < -1$, so when $-2 < x < -1$.

Similarly, $f'(x) < 0$ and $g'(x) > 0$ when $x < -2$ and either $x < -3$ or $x > -1$; together these give $x < -3$.

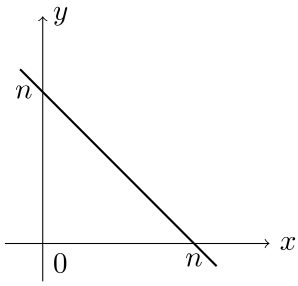
So the complete set of solutions (if we decide to include the points where $f'(x) = 0$) is $x \leq -3$ or $-2 \leq x \leq -1$, and so the answer is option E.

Question 17

It seems that we need to calculate $\mathrm{F}(n)$ before we can find $\mathrm{G}(n)$. We could do this either using integration or geometry. We start with an integration method:

$$\begin{aligned}\mathrm{F}(n) &= \frac{1}{n}\int_0^n (n-x)\,\mathrm{d}x \\ &= \tfrac{1}{n}\left[nx - \tfrac{1}{2}x^2\right]_0^n \\ &= \tfrac{1}{n}(n^2 - \tfrac{1}{2}n^2) \\ &= \tfrac{1}{2}n\end{aligned}$$

We could reach the same conclusion by sketching a graph of the function:



The area of this triangle is $\frac{1}{2}n^2$, so $\mathrm{F}(n) = \frac{1}{n} \times \frac{1}{2}n^2 = \frac{1}{2}n$

We now have:

$$\begin{aligned}\mathrm{G}(n) &= \sum_{r=1}^{n} \mathrm{F}(r) \\ &= \sum_{r=1}^{n} \tfrac{1}{2}r \\ &= \tfrac{1}{2}\sum_{r=1}^{n} r\end{aligned}$$

where for the last line we have used $(\frac{1}{2}\mathrm{F}(1) + \frac{1}{2}\mathrm{F}(2) + \cdots) = \frac{1}{2}(\mathrm{F}(1) + \mathrm{F}(2) + \cdots)$

Now $\sum_{r=1}^{n} r = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$, either using the formula for triangular numbers or of the sum of an arithmetic series.

So $\mathrm{G}(n) = \frac{1}{4}n(n+1)$.

Therefore

$$\begin{aligned} & \mathrm{G}(n) > 150 \\ \iff\quad & \tfrac{1}{4}n(n+1) > 150 \\ \iff\quad & n(n+1) > 600\end{aligned}$$

So we can try values: 25 is a nice number to calculate with, so we start with options C and D:

$$\begin{aligned}24 \times 25 &= 600 \\ 25 \times 26 &= 650\end{aligned}$$

so $n = 25$ is the smallest positive integer for which $\mathrm{G}(n) > 150$, and the answer is D.

Question 18

We can do this algebraically.

A translation of the graph of $y = \mathrm{f}(x)$ by 2 units in the positive y-direction gives the graph of $y = \mathrm{f}(x) + 2$

A stretch of factor k parallel to the x-axis gives the graph of $y = \mathrm{f}(\frac{1}{k}x) = \mathrm{f}(\frac{x}{k})$

For these to be the same, we need $\mathrm{f}(x) + 2 = \mathrm{f}(\frac{1}{k}x)$ for all (positive) x.

In our case, $\mathrm{f}(x) = \log_{10} x$, so we require

$$\log_{10} x + 2 = \log_{10}(\tfrac{x}{k})$$

But $\log_{10}(\frac{x}{k}) = \log_{10} x - \log_{10} k$, so we require $\log_{10} k = -2$, giving $k = 10^{-2} = 0.01$. Hence the answer is A.

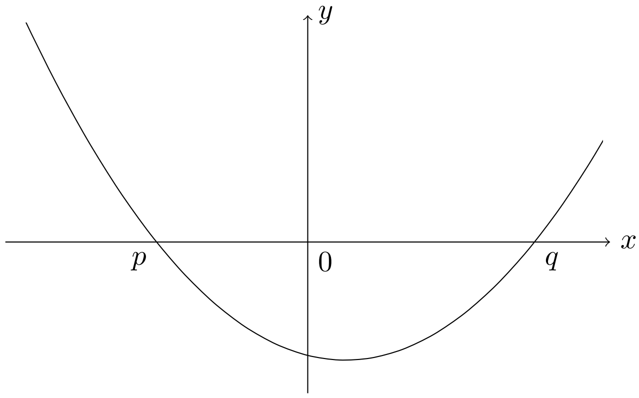
We have done more than required to answer this question: we have actually shown that these transformations are equivalent. An alternative approach is to assume that the question is correct and that these transformations are equivalent, and to use a single point to find the value of k.

We could see which point ends up on the x-axis after the translation: it was at $(a, -2)$, so it must have been at $(0.01, -2)$. Therefore the translated graph crosses the x-axis at 0.01, while the original graph crossed the x-axis at 1. Therefore the stretch needs to be by a factor of 0.01 parallel to the x-axis.

We could have done a similar calculation with any other point.

Question 19

We can sketch the graph of $y = x^2 + bx + c$. It intersects the x-axis at p and q, and intersects the y-axis at $c < 0$. So the graph must look like this:



However, the vertex might be either side of the y-axis.

To solve the inequality $x^2 + bcx + c^3 < 0$, we need to find where this equals zero. We can use the quadratic formula for this: $x^2 + bcx + c^3 = 0$ when

$$\begin{aligned} x &= \frac{-bc \pm \sqrt{(bc)^2 - 4c^3}}{2} \\ &= \tfrac{1}{2}(-bc \pm \sqrt{b^2c^2 - 4c^3}) \\ &= \tfrac{1}{2}c(-b \pm \sqrt{b^2 - 4c}) \end{aligned}$$

Going back to the original quadratic, we can solve it to find p and q in terms of b and c:

$$p, q = \tfrac{1}{2}(-b \pm \sqrt{b^2 - 4c})$$

Comparing these, we see that the roots of $x^2 + bcx + c^3$ are pc and qc in some order. So the solution is C or D.

To distinguish between these possibilities, we need to know which is the larger of pc and qc. Note that we were told that $c < 0$. Since $p < q$, it follows that $pc > qc$, and so the answer is D.

It turns out that we didn't actually need our graph sketch.

A somewhat less obvious approach to this question is as follows.

We can divide the second inequality by $c^2 > 0$, so $x^2 + bcx + c^3 < 0$ is equivalent to

$$\frac{x^2}{c^2} + \frac{bx}{c} + c < 0$$

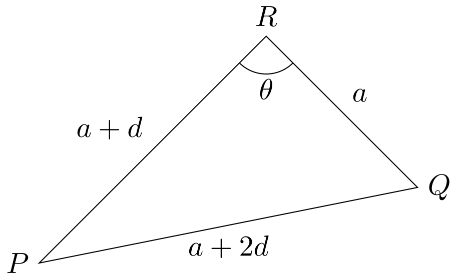
or

$$\left(\frac{x}{c}\right)^2 + b\left(\frac{x}{c}\right) + c < 0$$

This is the same as our original inequality, just with x replaced by $\frac{x}{c}$, so its solution is $p < \frac{x}{c} < q$. Multiplying this by $c < 0$ then gives the solution $pc > x > qc$.

Question 20

It would be really helpful to sketch this triangle so that we don't get confused! We will write θ for the angle $\angle PRQ$.



It is clear that we want to use the cosine rule to relate the three sides and the angle θ. Using '$c^2 = a^2 + b^2 - 2ab\cos C$', we have

$$(a + 2d)^2 = a^2 + (a + d)^2 - 2a(a + d)\cos\theta$$

which we can expand and simplify to give

$$a^2 + 4ad + 4d^2 = 2a^2 + 2ad + d^2 - 2a(a + d)\cos\theta$$

Rearranging gives

$$\cos\theta = \frac{a^2 - 2ad - 3d^2}{2a(a + d)}$$

We can simplify this by noticing that the numerator factorises:

$$\cos\theta = \frac{(a + d)(a - 3d)}{2a(a + d)} = \frac{a - 3d}{2a}$$

Since $3d > 2a$, $a - 3d < -a$, and so $\cos\theta < \frac{-a}{2a} = -\frac{1}{2}$. This gives $\theta > 120°$, and so the answer must be option E.

We can go further and show that $\theta < 180°$. Since PQR is a triangle, the length PQ must be less than the sum of PR and RQ, that is $a + 2d < a + d + a$, so $d < a$. This gives $\cos\theta > \frac{-2a}{2a} = -1$ so $\theta < 180°$.

An alternative approach is to just try values for d. We are given that $d > \frac{2}{3}a$, so we could try $d = \frac{2}{3}a$. This gives side lengths of a, $\frac{5}{3}a$, $\frac{7}{3}a$, which we can take to be 3, 5, 7 if we choose $a = 3$. Plugging these lengths into the cosine rule gives $\theta = 120°$, so this is one limit.

We are allowed larger values of d. If we take d too large, then we will no longer have a triangle, as $PQ > PR + RQ$. So as d increases, the angle θ must increase too. Therefore the answer must be E: $120° < \theta < 180°$.

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